

Approximate expressions for the beta-neutrino angular correlation coefficient

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Recently [1], an ‘exact’ calculation of the beta-neutrino angular correlation coefficient for ^{21}Na was published. The coefficient is defined as

$$a_{ev}(W) = a_{ev}^0 + \Delta a_{ev}(W) \quad (1)$$

with $a_{ev}^0 = (a_1^2 - \frac{1}{3}c_1^2)/(a_1^2 + c_1^2)$ being the major contribution, where $a_1 = g_V M_F$ and $c_1 = g_A M_{GT}$ with M_F and M_{GT} being the Fermi and Gamow-Teller matrix elements and g_V and g_A their respective coupling constants. Here W is the electron total energy expressed in electron rest-mass units. We computed the correction Δa_{ev} using the exact formalism of Behrens and Bühring (BB) [2] and found Δa_{ev} , when averaged over the electron energy spectrum, gave a correction of order 1%. Alternatively computing the correction with the formalism of Holstein [3] we found the correction to be much smaller, of order 0.1%. To try and resolve this discrepancy we have been working with the BB formalism, identifying the leading order terms and comparing them with those of Holstein.

The beta decay differential decay rate is written in Holstein [3] as

$$d^5\Gamma = \frac{G^2}{(2\pi)^5} F(Z, W)(W_0 - W)^2 p W dW d\Omega_e d\Omega_\nu \left(f_1(W) + f_2(W) \frac{p}{W} \hat{\mathbf{p}} \cdot \hat{\mathbf{k}} + \dots \right) \quad (2)$$

where $\hat{\mathbf{p}}$ and $\hat{\mathbf{k}}$ are unit vectors in the directions of the electron and neutrino respectively. Here W_0 is the maximum value of W , p is the electron momentum, with $p^2 = W^2 - 1$ in electron rest-mass units, and $F(Z, W)$ is the Fermi function. The beta-neutrino angular correlation coefficient is defined as

$$a_{ev} = \frac{f_2(W)}{f_1(W)} \quad (3)$$

In BB [2] the same decay rate is written

$$d^5\Gamma = \frac{G^2}{(2\pi)^5} F_0 L_0 (W_0 - W)^2 p W dW d\Omega_e d\Omega_\nu \left(C(W) + D(W) \frac{p}{W} \hat{\mathbf{p}} \cdot \hat{\mathbf{k}} + \dots \right) \quad (4)$$

and

$$a_{ev} = \frac{D(W)}{C(W)} \quad (5)$$

Here $F_0 = 2F(Z, W)/(1 + \gamma_1)$ with $\gamma_1 = (1 - (\alpha Z)^2)^{1/2}$; α is the fine-structure constant and Z is the charge number of the daughter nucleus, taken positive in electron decay and negative in positron decay.

Note the two expressions Eqs. (2) and (4) have different normalizations. This is because BB prefer to normalize their electron wave functions to the value obtained at the origin (L_0 represents the electron density at the origin), whereas Holstein uses the historical normalization at a radius R ($F(Z, W)$ represents the electron density at radius R). The relationship between the two is

$$f_1(W) = \frac{2}{(1+\gamma_1)} L_0 C(W) \quad f_2(W) = \frac{2}{(1+\gamma_1)} L_0 D(W) \quad (6)$$

If it is assumed that the electron moves in the Coulomb field of a uniform nuclear charge-density distribution of radius R , then to second order in (WR) and (αZ) , the relationship is [2]

$$\frac{2}{(1+\gamma_1)} L_0 = 1 - (\alpha Z)(WR) - \frac{1}{2} \frac{(\alpha Z)R}{W} + \frac{7}{15} (\alpha Z)^2, \quad (7)$$

where R is given in electron Compton wavelength units. Note that the dimensionless R is a small quantity, such that (WR) and (αZ) can be considered expansion parameters.

In Holstein's formalism all the aspects of nuclear-structure physics are encoded in terms of a few form factors, $a(q^2)$, $b(q^2)$, $c(q^2)$, $d(q^2)$, $h(q^2)$. . . , where q^2 is the square of the four-momentum transfer. On invoking the impulse approximation these can be related to nuclear-structure matrix elements calculable in the shell model. The important ones are: $M_F = \langle 1 \rangle$, $M_{r^2} = \langle r^2 \rangle \cong \frac{3}{5} R^2 \langle 1 \rangle$, $M_{GT} = \langle \sigma \rangle$, $M_{\sigma r^2} = \langle r^2 \sigma \rangle \cong \frac{3}{5} R^2 \langle \sigma \rangle$, $M_{1y} = (16\pi/5)^{1/2} \langle r^2 [Y_2 \times \sigma] \rangle$ and $M_L = \langle L \rangle$. It is convenient to express M_{1y} in terms of M_{GT} (valid only for allowed transitions for which M_{GT} does not vanish) by defining x as

$$x = -\sqrt{10} \frac{M_{1y}}{M_{\sigma r^2}} \quad (8)$$

Our goal is to write the spectral functions $f_1(W)$ and $f_2(W)$ introduced in Eq. (2) in terms of the four parameters: a_1 , c_1 , the weak-magnetism form factor b , and x . For the present time, we have left out the relativistic matrix elements denoted in [3] as $M_{r,p}$ and $M_{\sigma r,p}$, while the matrix element $M_{\sigma L}$ vanishes in diagonal matrix elements as would occur in a mirror transition between isobaric analogue states. We have also dropped the small pseudoscalar term in $h(q^2)$ and the second-class current term in $d(q^2)$.

In the method of Behrens and Bühring [2], all beta-decay observables are given in terms of quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$, which are linear combinations of form factor coefficients F_{KLS} and electron and neutrino radial wave functions. To the extent that these radial wave functions can be computed exactly, by solving the appropriate Dirac equation, the formalism can be considered exact. Here K is the multipolarity of the transition, L and s are the orbital and spin quantum numbers characterising the transition, and k_e and k_ν are the partial-wave counting indices for the electron and neutrino radial functions. For allowed transitions, it is usually sufficient to consider the two lowest partial waves, $k_{e,\max} = 2$, $k_{\nu,\max} = 2$. The strength of BB's work, however, is that they give expansions of the electron and neutrino wave functions in power series of (αZ) , (WR) and $(p_\nu R)$, where W and $p_\nu =$

$W_0 - W$ are the electron and neutrino energies in electron rest-mass units. These expansions should enable us to make contact with the expressions of Holstein. We will quote formulae to second order in the quantities.

The form factor coefficients, F_{KLs} , are very similar to Holstein's form factors and by invoking the impulse approximation they too can be related to nuclear-structure matrix elements. To show the relationships between Holstein and BB's expressions it is convenient to separate the Fermi and Gamow-Teller pieces, $f_1(W) = f_1^F(W) + f_1^{GT}(W)$ and $f_2(W) = f_2^F(W) + f_2^{GT}(W)$. Further, we display their electron energy dependence explicitly by writing them as

$$f_\alpha^F(W) = a_1^2 k_\alpha^F (1 + A_\alpha^F W + \frac{B_\alpha^F}{W} + C_\alpha^F W^2)$$

$$f_\alpha^{GT}(W) = c_1^2 k_\alpha^{GT} (1 + A_\alpha^{GT} W + \frac{B_\alpha^{GT}}{W} + C_\alpha^{GT} W^2), \alpha = 1, 2 \quad (9)$$

Expressions for the parameters of these expansions are given in Table I for $f_1(W)$ and in Table II for $f_2(W)$.

Table I. Expressions for the spectral function $f_1(W)$ derived from the formulations of Holstein and Behrens-Bühring.

	Holstein	Behrens-Bühring
Fermi terms		
k_1	$1 - \frac{1}{5}(W_0 R)^2 + \frac{1}{15}R^2 - \frac{6}{35}(\alpha Z)(W_0 R)$	$1 - \frac{1}{5}(W_0 R)^2 + \frac{1}{15}R^2 - \frac{6}{35}(\alpha Z)(W_0 R)$
$k_1 A_1$	$\frac{4}{15}(W_0 R)R - \frac{48}{35}(\alpha Z)R$	$\frac{4}{15}(W_0 R)R - \frac{48}{35}(\alpha Z)R$
$k_1 B_1$	$\frac{2}{15}(W_0 R)R - \frac{18}{35}(\alpha Z)R$	$\frac{2}{15}(W_0 R)R - \frac{18}{35}(\alpha Z)R$
$k_1 C_1$	$-\frac{4}{15}R^2$	$-\frac{4}{15}R^2$
Gamow-Teller terms		
k_1	$1 - \frac{1}{5}(W_0 R)^2 + \frac{11}{45}R^2(1 - \frac{1}{10}x) + \frac{2}{35}(\alpha Z)(W_0 R) - \frac{2}{3}(W_0 R)\frac{b}{Ac_1}\frac{1}{MR}$	$1 - \frac{1}{5}(W_0 R)^2 + \frac{11}{45}R^2(1 - \frac{2}{11}x) + \frac{2}{35}(\alpha Z)(W_0 R)(1 - x) - \frac{2}{3}(W_0 R)\frac{b}{Ac_1}\frac{1}{MR} + \frac{22}{25}(\alpha Z)\frac{b}{Ac_1}\frac{1}{MR}$
$k_1 A_1$	$\frac{4}{9}(W_0 R)R(1 - \frac{1}{10}x) - \frac{8}{5}(\alpha Z)R + \frac{4}{3}R\frac{b}{Ac_1}\frac{1}{MR}$	$\frac{4}{9}(W_0 R)R(1 - \frac{1}{10}x) - \frac{8}{5}(\alpha Z)R(1 - \frac{1}{28}x) + \frac{4}{3}R\frac{b}{Ac_1}\frac{1}{MR}$
$k_1 B_1$	$-\frac{2}{45}(W_0 R)R(1 - \frac{11}{20}x) - \frac{18}{35}(\alpha Z)R - \frac{2}{3}R\frac{b}{Ac_1}\frac{1}{MR}$	$-\frac{2}{45}(W_0 R)R(1 - x) - \frac{18}{35}(\alpha Z)R - \frac{2}{3}R\frac{b}{Ac_1}\frac{1}{MR}$
$k_1 C_1$	$-\frac{4}{9}R^2(1 - \frac{1}{10}x)$	$-\frac{4}{9}R^2(1 - \frac{1}{10}x)$

Table II. Expressions for the spectral function $f_2(W)$ derived from the formulations of Holstein and Behrens-Bühring.

	Holstein	Behrens-Bühring
Fermi terms		
k_2	$1 - \frac{1}{5}(W_0R)^2 + \frac{1}{5}R^2 - \frac{6}{35}(\alpha Z)(W_0R)$	$1 - \frac{1}{5}(W_0R)^2 + \frac{1}{5}R^2 - \frac{6}{35}(\alpha Z)(W_0R)$
k_2A_2	$-\frac{48}{35}(\alpha Z)R$	$-\frac{48}{35}(\alpha Z)R$
k_2B_2	0	$-\frac{1}{2}(\alpha Z)R$
k_2C_2	0	0
Gamow-Teller terms		
k_2	$-\frac{1}{3} + \frac{1}{15}(W_0R)^2(1 + \frac{1}{5}x) - \frac{1}{15}R^2(1 - \frac{1}{10}x) - \frac{6}{35}(\alpha Z)(W_0R) + \frac{2}{3}(W_0R)\frac{b}{Ac_1}\frac{1}{MR}$	$-\frac{1}{3} + \frac{1}{15}(W_0R)^2(1 - \frac{2}{5}x) - \frac{1}{15}R^2(1 - \frac{2}{5}x) - \frac{6}{35}(\alpha Z)(W_0R) + \frac{2}{3}(W_0R)\frac{b}{Ac_1}\frac{1}{MR} - \frac{22}{75}(\alpha Z)\frac{b}{Ac_1}\frac{1}{MR}$
k_2A_2	$-\frac{8}{15}(W_0R)R(1 - \frac{1}{10}x) + \frac{24}{35}(\alpha Z)R - \frac{4}{3}R\frac{b}{Ac_1}\frac{1}{MR}$	$-\frac{8}{15}(W_0R)R(1 - \frac{13}{180}x) - \frac{68}{105}(\alpha Z)R(1 + \frac{61}{1190}x) - \frac{4}{3}R\frac{b}{Ac_1}\frac{1}{MR}$
k_2B_2	0	$-\frac{1}{2}(\alpha Z)R$
k_2C_2	$\frac{8}{15}R^2(1 - \frac{1}{10}x)$	$\frac{8}{15}R^2(1 - \frac{13}{180}x)$

For the Fermi terms, it is noted there is almost complete agreement between the Holstein and BB formalisms; the only difference is in a small B_2 coefficient in $f_2(W)$. For the Gamow-Teller terms, we note the following:

- There are no electromagnetic (αZ) corrections to the weak-magnetism terms, b , in coefficients k_1 and k_2 ;
- The coefficient of the matrix-element ratio, x (see Eq. (8)) is in many, but not all, instances different.

These differences do not represent errors, but reflect different approximations made in the derivations. There is little numerical consequence from these differences. The correction to the beta-neutrino correlation coefficient, Δa_{ev} , is likely to be small and of order 0.1% from these formulae.

However, we have not yet included the relativistic terms: the matrix elements $M_{r,p}$ and M_{orp} . These could lead to terms of comparable size to the terms considered so far. We also need to do some numerical studies on the exact formulation to understand their differences from the approximate formulae. Further study is underway.

- [1] V.E. Iacob, J.C. Hardy, C.A. Gagliardi, J. Goodwin, N. Nica, H.I. Park, G. Tabacaru, L. Trache, R.E. Tribble, Y. Zhai and I.S. Towner, *Phys. Rev. C* **74**, 015501 (2006).
- [2] H. Behrens and W. Bühring, *Electron Radial Wave Functions and Nuclear Beta-decay* (Clarendon Press, Oxford, 1982); H. Behrens, H. Genz, M. Conze, H. Feldmeir, W. Stock and A. Richter, *Ann. of Phys.* **115**, 276 (1978).
- [3] B.R. Holstein, *Rev. Mod. Phys.* **46**, 789 (1974).